Modelling the motion of a point-sized aircraft in response to a vortex tube encounter

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Abstract

Start from previous paper [N.T.Basse, Modelling of vortex-induced aviation turbulence, Meteorol. Atmos. Phys. Vol. ?, pp. ?-?, 2019]. Build model where vortex tube is much larger than aircraft, so aircraft can be modelled as a point. Simple equations of motion using aircraft acceleration due to vortex tube. Study velocity and position of aircraft. We find that mechanism is amplification of gravity wave by vortex tube.

Keywords: Aviation turbulence, Vortex tube, Aircraft motion, Gravity wave

1. Introduction

Refer to [1], where aircraft acceleration due to various vortex tube (VT) sizes was modelled.

Here, we consider the aircraft as being a point, i.e. that the dimensions of the VT are much larger than the aircraft. This simplifies the model.

Goal: Simple model to predict position/velocity and other measurable quantities.

The paper is organized as follows:

2. Model

2.1. Aircraft geometry

The wing and fuselage areas are defined as rectangular areas:

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$$A_{\text{fuselage}} = LH \tag{1}$$
$$A_{\text{wing}} = CS, \tag{2}$$

where L is the fuselage length, H is the fuselage height, C is the wing chord and S is the wing span.

As in [1], we use dimensions similar to those of an Airbus A330-200 [2], i.e. H=C=6 m and L=S=60 m.

2.2. Physical constants

The specific gas constant for dry air is [3]:

$$R_{\text{specific,air}} = 287.058 \text{ J}/(\text{kg} \cdot \text{K}) \tag{3}$$

The polytropic index is [4]:

$$n = 1.235 \tag{4}$$

The density of air at sea level is [3]:

$$\rho_0 = 1.225 \text{ kg/m}^3 \tag{5}$$

The pressure of air at sea level is [3]:

$$p_0 = 101325 \text{ Pa} \tag{6}$$

The temperature of air at sea level is [3]:

$$T_0 = 288.15 \text{ K} \tag{7}$$

The Earth mean radius is [5]:

$$r_{\text{Earth}} = 6.371 \times 10^6 \text{ m} \tag{8}$$

The Earth mass is [5]:

$$m_{\text{Earth}} = 5.9722 \times 10^{24} \text{ kg} \tag{9}$$

The gravitational constant is [6]:

$$G = 6.67384 \times 10^{-11} \text{ m}^3/(\text{kg} \cdot \text{s}^2) \tag{10}$$

2.3. Initialisation

Max thrust at sea level:

$$\mathcal{T}_0 = 600 \times 10^3 \text{ N} \tag{11}$$

Thrust factor (50%):

$$\mathcal{T}_{\text{factor}} = 0.5 \tag{12}$$

Initial altitude:

$$z_{\text{init}} = 10^4 \text{ m} \tag{13}$$

Initial speed (in x-direction):

$$v_{\text{init}} = 800/3.6 = 222.2 \text{ m/s} \tag{14}$$

Initial aircraft mass:

$$m_{\text{init}} = 230 \times 10^3 \text{ kg} \tag{15}$$

Fuel consumption:

$$F = 6/1000 \text{ kg/m} \tag{16}$$

The initial gravitational acceleration is [7]:

$$g_{\text{init}} = \frac{G m_{\text{Earth}}}{(r_{\text{Earth}} + z_{\text{init}})^2} \tag{17}$$

The acceleration of a vortex tube event:

$$a_{\text{VT}} = 0.5 \times g_{\text{init}} \tag{18}$$

Determine viscous damping coefficient value from:

$$c1 = \frac{F}{v} = \frac{m \cdot a}{v} \tag{19}$$

Using $m = m_{\text{init}}$, $a = a_{\text{VT}}$ and $v = v_{\text{init}}$, we set the viscous damping coefficient in the y- and z-direction equal:

$$c1_y = c1_z = 5000[CHECK!] \text{ kg/s} \tag{20}$$

The initial density of air is [4]:

$$\rho_{\text{init}} = \rho_0 \times \left(1 - \frac{n-1}{n}\frac{\rho_0 g_{\text{init}}}{p_0} z_{\text{init}}\right)^{1/(n-1)} \tag{21}$$

The initial air temperature is [4]:

$$T_{\text{init}} = T_0 \times \left(1 - \frac{n-1}{n}\frac{\rho_0 g_{\text{init}}}{p_0} z_{\text{init}}\right) \tag{22}$$

The initial dynamic viscosity of air is derived from the initial temperature and density using a real gas model [8].

The maximum initial thrust is:

$$\mathcal{T}_{\text{init,max}} = \mathcal{T}_0 * \rho_{\text{init}}/\rho_0 \tag{23}$$

At the initial altitude, the 50% thrust is:

$$\mathcal{T}_{\text{init,50}} = \mathcal{T}_{\text{factor}} \times \mathcal{T}_{\text{init,max}} = 0.5 \times \mathcal{T}_{\text{init,max}} \tag{24}$$

The drag (C_d) and lift (C_l) coefficients are constants based on initial conditions. Here, (i) thrust balances drag and (ii) gravity balances lift:

$$C_d = \frac{2\mathcal{T}_{\text{init,50}}}{\rho_{\text{init}} v_{\text{init}}^2 A_{\text{wing}}} \tag{25}$$

$$C_l = \frac{2 m_{\text{init}} g_{\text{init}}}{\rho_{\text{init}} v_{\text{init}}^2 A_{\text{wing}}} \tag{26}$$

2.4. Vortex tube case

The angular velocity of the VT is:

$$\Omega = a_{\text{VT}}/v_{\text{init}} \tag{27}$$

The vorticity of the VT is [1]:

$$\omega_x = -2\Omega \sin(\theta)\cos(\phi) \tag{28}$$

$$\omega_y = -2\Omega \sin(\theta)\sin(\phi) \tag{29}$$

$$\omega_z = -2\Omega \cos(\theta), \tag{30}$$

where θ is the polar angle and ϕ is the azimuthal angle.

2.5. Equations of motion

Two time loops:

1. First loop to establish stable altitude
2. Second loop which includes the VT encounter

Time step size Δt of 0.1 s. Smaller steps lead to more precise results, but take longer time.

Length of time for loops is 1900 and 2100 s (may change).

The aircraft mass is calculated as:

$$m(t) = m_{\text{init}} - F\left(x(t) - x_{\text{init}}\right) \tag{31}$$

The drag and lift coefficients are:

$$A_d(t) = \frac{C_d \rho(t) A_{\text{wing}}}{2m(t)} \tag{32}$$

$$A_l(t) = \frac{C_l \rho(t) A_{\text{wing}}}{2m(t)} \tag{33}$$

The thrust is corrected because of the reduction of aircraft mass:

$$\mathcal{T}_{\text{corr}} = \frac{m(t)}{m_{\text{init}}} \tag{34}$$

This means the thrust becomes:

$$\mathcal{T}(t) = \mathcal{T}_{\text{corr}} \times \mathcal{T}_{\text{factor}} \times \mathcal{T}_0 \times \frac{\rho(t)}{\rho_0} \tag{35}$$

2.5.1. Outside vortex tube

Cartesian coordinate system:

1. x-direction is longitudinal
2. y-direction is transverse
3. z-direction is vertical

Acceleration in longitudinal (x), transverse (y) and vertical (z) directions:

$$a_x(t) = \mathcal{T}(t)/m(t) - A_d(t) \times v_x(t)^2 \tag{36}$$

$$a_y(t) = -c1_y v_y(t)/m(t) \tag{37}$$

$$a_z(t) = A_l(t) \times v_x(t)^2 - g(t) - c1_z \times v_z(t)/m(t) \tag{38}$$

2.5.2. Inside vortex tube

Acceleration in longitudinal (x), transverse (y) and vertical (z) directions:

$$\begin{aligned} a_x(t) &= \mathcal{T}(t)/m(t) - A_d(t) \times v_x(t)^2 \\ &\quad +1/2\left(\omega_y v_z(t) - \omega_z v_y(t)\right) \qquad (39) \\ a_y(t) &= -c1_y v_y(t)/m(t) \\ &\quad +1/2\left(\omega_z v_x(t) - \omega_x v_z(t)\right) \qquad (40) \\ a_z(t) &= A_l(t) \times v_x(t)^2 - g(t) - c1_z \times v_z(t)/m(t) \\ &\quad +1/2\left(\omega_x v_y(t) - \omega_y v_x(t)\right) \qquad (41) \end{aligned}$$

2.6. Time stepping

(Forward) Euler scheme

Velocity at timestep $t + \Delta t$:

$$\begin{aligned} v_x(t+\Delta t) &= v_x(t) + a_x(t)\Delta t \qquad (42) \\ v_y(t+\Delta t) &= v_y(t) + a_y(t)\Delta t \qquad (43) \\ v_z(t+\Delta t) &= v_z(t) + a_z(t)\Delta t \qquad (44) \end{aligned}$$

Position at timestep $t + \Delta t$:

$$\begin{aligned} x(t+\Delta t) &= x(t) + v_x(t)\Delta t \qquad (45) \\ y(t+\Delta t) &= y(t) + v_y(t)\Delta t \qquad (46) \\ z(t+\Delta t) &= z(t) + v_z(t)\Delta t \qquad (47) \end{aligned}$$

Gravitational acceleration at timestep $t + \Delta t$:

$$g(t+\Delta t) = \frac{G m_{\text{Earth}}}{(r_{\text{Earth}} + z(t+\Delta t))^2} \qquad (48)$$

Density at timestep $t + \Delta t$:

$$\rho(t+\Delta t) = \rho_0 \times \left(1 - \frac{n-1}{n}\frac{\rho_0 g(t+\Delta t)}{p_0} z(t+\Delta t)\right)^{1/(n-1)} \qquad (49)$$

Temperature at timestep $t + \Delta t$:

$$T(t+\Delta t) = T_0 \times \left(1 - \frac{n-1}{n}\frac{\rho_0 g(t+\Delta t)}{p_0} z(t+\Delta t)\right) \qquad (50)$$

Dynamic viscosity at timestep $t+\Delta t$ is derived from $\rho(t+\Delta t)$ and $T(t+\Delta t)$.

2.7. Postprocessing

2.7.1. Energy [MATLAB 1/4]

is this section really necessary, what is the purpose?

calculate energy loss due to: drag (x), lift (z), viscous damping (y and z):

$$W = \int F \mathrm{d}s, \tag{51}$$

where s in our case can be either x, y or z.

F is previously defined.

2.7.2. Speed of sound [MATLAB 2/4]

sos: dry air prop to sqrt temp, use to construct Mach no

$$c_{\text{ideal}} = \sqrt{\gamma \frac{p}{\rho}}, \tag{52}$$

where $\gamma = 1.4$ for air.

Calculate Mach number.

3. Results

3.1. Vortex tube geometry

VTs in the three principal directions.

Size is defined to have a vortex/aircraft area ratio $r_A = 10$, which means that the vortex tube is large compared to the aircraft area [1]:

- Vortex radius $R = 33.9$ m
- Vortex width $W = 53.2$ m

Cases are in Table 1.

Table 1: Table of cases. Case 1 is the baseline case.

Case no	ϕ [rad]	θ [rad]	y_0 [m]	z_0 [m]	R [m]	W [m]
1	$\pi/2$	$\pi/2$	0	0	33.9	53.2
18	$\pi/2$	π	0	0	33.9	53.2
19	π	$\pi/2$	0	0	33.9	53.2

3.2. Transverse (horizontal) vortex tube (case 1)

transverse vortex tube (TVT, corresponds to what is usually called horizontal vortex tube (HVT)

3.2.1. No viscous damping

c1_z=0; no displacement in z-direction, but oscillation around mean continuously after VT. find max velocity in z-direction, around 1.4 m/s. oscillating continuously after VT.

Link to atmospheric (pure) gravity waves [9]. Stratified fluid, i.e. density changing with altitude.

Brunt–Väisälä (B-V) angular frequency:

$$\omega_{\mathrm{B-V}} = \sqrt{\frac{g}{\theta}\frac{\mathrm{d}\theta}{\mathrm{d}z}}, \tag{53}$$

where θ is the potential temperature:

$$\theta = T\left(\frac{p_0}{p}\right)^{R/c_p} \tag{54}$$

and R/c_p=0.286 for air.

We find that:

$$T_{\mathrm{B-V}} = \frac{2\pi}{\omega_{\mathrm{B-V}}} \tag{55}$$

is proportional to the modelled oscillation period. Calibrated at our standard case we find:

$$T_{\mathrm{Model\ 1}} = \frac{T_{\mathrm{B-V}}}{2.893} \tag{56}$$

Another approach is to set a reference potential temperature θ_0 [9]:

$$\omega_{\mathrm{B-V,ref}} = \sqrt{\frac{g}{\theta_0}\frac{\mathrm{d}\theta}{\mathrm{d}z}}, \tag{57}$$

where θ_0=39.15 K when calibrating to our standard case. Here:

$$T_{\mathrm{Model\ 2}} = \frac{2\pi}{\omega_{\mathrm{B-V,ref}}} \tag{58}$$

Which model is best (smallest root-mean-square deviation), Model 1 or 2?

3.2.2. With viscous damping

c1_z=5000[CHECK!]; z-position: three peaks visible (same as velocity), back to before VT. 22 m change for first peak. single velocity peak (1.5 m/s), second and third peaks visible

3.3. Case 18: Vertical vortex tube

$\theta = \pi$ (see case 17 in [1])

c1_y=0; y-position continually increases max velocity in y-direction, around 1.5 m/s. stays constant after reaching max.

c1_y=5000[CHECK!]; 41 m y-displacement single velocity peak (1.5 m/s), no oscillation

3.4. Case 19: Longitudinal vortex tube

$\phi = \pi$ (see case 15 in [1])

cases:

no damping: y continually increasing, z oscillating z/v_z/a_z all oscillating all the time

fixed damping (5000[CHECK!]): y becomes constant no oscillation in z

4. Discussion

4.1. Richardson number [MATLAB 3/4]

Richardson number [10]:

$$R_i = \frac{\omega_{\mathrm{B-V}}^2}{\left(\partial v_{\mathrm{hor}}/\partial z\right)^2}, \tag{59}$$

where $v_{\mathrm{hor}} = \sqrt{v_x^2 + v_y^2}$ is the horizontal velocity and $\partial v_{\mathrm{hor}}/\partial z$ is the vertical shear. If $R_i < 0.25$ it is theoretically predicted that the shear instability occurs.

Acceleration due to vortex tube leads to larger vertical shear? If yes, could explain excitation.

4.2. Energy transfer

`https://physicstoday.scitation.org/doi/10.1063/PT.3.4225`
(see papers in folder)

4.3. Phugoid mode

Oscillation is called the phugoid mode - from [11], the oscillation period is:

$$T_{\mathrm{ph}} = \frac{\sqrt{2}\pi v_x}{g} \tag{60}$$

This mode is very lightly damped, the damping ratio is:

$$\zeta = \frac{1}{\sqrt{2}}\frac{C_d}{C_l} \tag{61}$$

Scaling of the oscillation frequency

Modification of phugoid oscillation period, fits without velocity dependency:

$$T_{\mathrm{ph,1p}} = \frac{a_{1p}}{g}, \tag{62}$$

where a_{1p}=1868 m/s, fmin=14.8. Calibrating at 1 g yields a constant which is 1782 m/s.

$$T_{\mathrm{ph,2p}} = \frac{a_{2p}}{g^{b_{2p}}}, \tag{63}$$

where a_{2p}=2558, b_{2p}=1.2 and fmin=2.6. Here, we can not apply units to the fit parameters since b_{2p} is not an integer.

Also a small dependency of oscillation period on altitude z.

No dependency: Speed, weight, wing area, thrust etc.

Conclusion: This is not the phugoid mode. Can it be that sometimes the phugoid mode is mistaken for the gravity wave?

4.4. Sound [MATLAB 4/4]

Link to (linear theory of) vortex-airfoil interaction noise [12, 13, 14]. This is about acoustic pressure, we study pressure at the aircraft altitude. Compare pressure signatures for the two cases.

Sound generated by aircraft in response to vortex tube encounter [13]:

$$\mathcal{S} = \frac{1}{\rho(t)} \nabla \cdot (\rho(t) \boldsymbol{\omega} \times \mathbf{v}) \tag{64}$$

Or - for low Mach number:

$$\mathcal{S} = \nabla \cdot (\boldsymbol{\omega} \times \mathbf{v}) \tag{65}$$

5. Conclusions

Aircraft encounter with VT amplifies gravity waves.

Explanation: Sudden acceleration due to vortex tube increases vertical shear, which decreases the Richardson number, thus destabilizing the gravity wave.

Next: (i) Comparison to measurements and (ii) model which includes smaller VTs, i.e. a combination of [1] and this paper.

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